Int. J. Heat Mass Transfer. Vol. 30, No. 7, pp. 1259–1267, 1987
Printed in Great Britain

0017-9310/87 \$3.00+0.00
Pergamon Journals Ltd.

Solution of the unsteady-state heat conduction problem for a two-dimensional region with a moving boundary

N. M. TSIRELMAN and A. V. ZHIBER
The Sergo Ordzhonikidze Ufa Aviation Institute, 450000, Ufa, U.S.S.R.

(*Received* 18 *September* 1985)

Abstract—With the use of the convolution-type functional a variational description is given for the process of unsteady-state heat conduction with the first-kind boundary conditions for a two-dimensional region whose boundary moves in time according to the familiar arbitrary law. Based on the Galerkin–Kantorovich method, a corresponding system of Euler equations is written the solution of which (numerical or analytical) is required to determine the temperature field in each specific case. As an example, the first and second analytic approximations to the solution of the above problem are obtained for the case of the deformation of a prism having initially a circular cross-section.

INTRODUCTION

THE DETERMINATION of temperature fields in bodies whose size and shape vary in time is an important problem of the technological thermal physics when consideration is given to the treatment of metals and alloys by traditional techniques (plastic metal working, machining, grinding, etc.). The solution of this problem is also required when account is made of the abrasion in time of thermally stressed heat engine elements, evaporation of liquid droplets in a gas flow, etc. The same problems, but in a different terminology, are encountered, for example, in the theory of strength, in electrodynamics and filtration.

A change in the shape of the body and in the motion of its boundary leads to a situation requiring that the classical linear heat conduction theory methods (the separation of variables, integral transformations, etc.) be preliminarily subjected to special transformations, a detailed description of which is given in ref. [1]. Note that the first results associated with a moving boundary seem to be those obtained by Lyubov [2]. Later, Grinberg [3] obtained a functional transformation which converts the boundary-value problem studied in such a moving coordinate system in which the transformed heat conduction equation admitted an exact solution by separating the variables over a segment for certain laws of the motion of a boundary and corresponding conditions on it.

Kartashov and Nechayev [4] developed the method of construction of Green's functions in non-cylindrical regions and illustrated its effectiveness over a segment for uniform motion of one of the boundaries and assignment of the first-kind boundary conditions.

The mathematical aspects of the heat conduction boundary-value problem in the region with a moving boundary and some methods of its numerical and analytical solution are discussed elsewhere [5]. In all of these methods [2–5] the thermophysical properties of the body material are assumed to be constant.

Based on the variational description of the phenomenon, studied with the use of the convolution-type functional, the method of constructing an approximate analytical solution to the heat conduction problem over a segment in the case of an arbitrary law of boundary motion and arbitrary boundary conditions for the space- and time-dependent thermophysical characteristics of the medium was for the first time developed in work [6].

It should be noted that despite the requirements of practice, the literature lacks any exact or approximate analytical solution to the unsteady-state heat conduction problem in a two-dimensional region with a moving outer boundary. This is due, of course, to the great difficulty of obtaining such a solution.

In the present work, which extends the results obtained in ref. [6], the method has been developed for obtaining an approximate analytical solution of the above-mentioned two-dimensional problem for an arbitrary law of body boundary motion, and an example of its application is given.

STATEMENT OF THE PROBLEM AND ITS REDUCTION TO A CYLINDRICAL REGION

Consider, in the rectangular system of coordinates x, y, τ, the region Q_t bounded from above by the figure $\Omega(t)$ on the plane $\tau = t$, from below by the figure $\Omega(0)$ on the plane $\tau = 0$ and from the side by the surface S_t (Fig. 1). The formation of the region Q_t corresponds to the arbitrary transition of the figure $\Omega(0)$ into the figure $\Omega(t)$ on the plane (x, y) on the time interval $[0, t]$.

Let $T(x, y, \tau)$ be the solution of the following two-dimensional boundary unsteady-state heat conduction problem with a moving external boundary

NOMENCLATURE

$c\rho(x,y,\tau)$ volumetric heat capacity of the body material
$q(x,y,\tau)$ power of volumetric heat generation sources
$T(x,y,\tau)$ temperature of the point with coordinates x, y at time τ
$T_0(x,y)$ function of the initial temperature distribution
$T_w(x,y,\tau)$ temperature on the surface bounding a body.

Greek symbols
$\lambda(x,y,\tau)$ thermal conductivity of the body material.

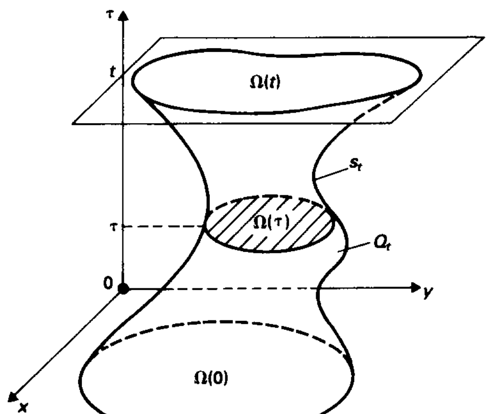

FIG. 1. Non-cylindrical region of the developing unsteady-state heat conduction process.

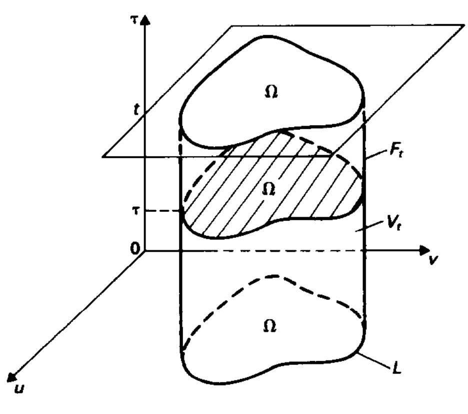

FIG. 2. Cylindrical region of the developing unsteady-state heat conduction process.

$$c\rho(x,y,\tau)T_\tau = \operatorname{div}[\lambda(x,y,\tau)\nabla T] + q(x,y,\tau), \quad (x,y,\tau)\in Q_t \quad (1)$$

$$T(x,y,0) = T_0(x,y), \quad (x,y)\in\Omega(0) \quad (2)$$

$$T(x,y,\tau) = T_w(x,y,\tau), \quad (x,y,\tau)\in S_t. \quad (3)$$

Here $c\rho$, λ, q and T_w are the prescribed functions of the variables x, y, τ, and $T_0(x,y)$ is the function of the variables x, y. Further, suppose there is also another space with the system of coordinates u, v, τ and with the cylindrical region $V_t = \{(u,v,\tau): (u,v)\in\Omega, \tau\in(0,t)\}$ (Fig. 2). Assume that the regions Q_t and V_t are in one-to-one continuous correspondence brought about by the formulae

$$\left.\begin{aligned} x &= x(u,v,\tau) \\ y &= y(u,v,\tau) \\ \tau &= \tau. \end{aligned}\right\} \quad (4)$$

In this case, to the points of the upper and lower bases Ω and of the side surface F_t of the cylinder V_t there respectively correspond the points of the surfaces $\Omega(t)$, $\Omega(0)$ and S_t that bound the region Q_t and conversely formulae (4) yield the relations

$$\left.\begin{aligned} u &= u(x,y,\tau) \\ v &= v(x,y,\tau) \\ \tau &= \tau. \end{aligned}\right\} \quad (5)$$

Assume that functions u and v, defined by formulae (5), have continuous first-order derivatives in the variable τ and continuous partial derivatives in the variables x and y up to second order inclusive. In the new variables u, v, τ the problem (1)–(3) will be stated as

$$c\rho T_\tau = a_{11}T_{uu} + a_{12}T_{uv} + a_{22}T_{vv} + a_1 T_u + a_2 T_v + q, \quad (u,v,\tau)\in V_t \quad (6)$$

$$T = T_0, \quad (u,v,0)\in\Omega \quad (7)$$

$$T = T_w, \quad (u,v,\tau)\in F_t. \quad (8)$$

Here, the old notation was used, namely, the function $c\rho = c\rho(u,v,\tau)$ is understood to be the function $c\rho(x(u,v,\tau), y(u,v,\tau),\tau)$, $T = T(u,v,\tau)$ denotes the function $T(x(u,v,\tau), y(u,v,\tau),\tau)$ and so on. The functions $a_{ij}, a_i, i,j = 1,2$ are calculated from the following formulae with the aid of relations (4):

$$a_{11} = \lambda\nabla^2 u, \quad a_{12} = 2\lambda\nabla u\nabla v, \quad a_{22} = \lambda\nabla^2 v,$$

$$a_1 = \operatorname{div}\lambda\nabla u - c\rho u_\tau, \quad a_2 = \operatorname{div}\lambda\nabla v - c\rho v_\tau.$$

THE CONSTRUCTION OF THE CONVOLUTION-TYPE FUNCTIONAL

Construct the functional $J(T)$ in such a way that the solution of problem (6)–(8) could be its stationary point and, consequently, could transform the first variation $J(T)$ to zero, i.e. $\delta J(T) = 0$. For convenience, adopt the following notation:

$$f(\tau) = f(u,v,\tau), \quad f = f(u,v).$$

The functional $J(T)$ will be sought in the form

$$J(T) = \int_0^t \iint_\Omega [R(\tau)T_\tau(\tau)+A_{11}(\tau)T_{uu}(\tau) + A_{12}(\tau)T_{uv}(\tau)+A_{22}(\tau)T_{vv}(\tau) + A_1(\tau)T_u(\tau)+A_2(\tau)T_v(\tau) + A(\tau)T(\tau)+Q(\tau)]T(t-\tau)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau + \iint_\Omega R(t)[T(0)-2T_0]T(t)\,\mathrm{d}u\,\mathrm{d}v + \int_0^t \int_L \{[q_1(\tau)T(t-\tau)+q_2(\tau)T_u(t-\tau) + q_3(\tau)T_v(t-\tau)][T(\tau)-T_w(\tau)] + q_4(\tau)T(\tau)T_u(t-\tau) + q_5(\tau)T(\tau)T_v(t-\tau)\}\,\mathrm{d}l\,\mathrm{d}\tau \tag{9}$$

where L is the curve bounding the plane region Ω.

Assume $J(T) = J_1+J_2+J_3$, where J_i $(i = 1,2,3)$ denote the integrals entering into the functional $J(T)$, respectively. Calculate the first variation of the functional J_1

$$\delta J_1 = \int_0^t \iint_\Omega [R(\tau)T_\tau(\tau)+A_{11}(\tau)T_{uu}(\tau) + A_{12}(\tau)T_{uv}(\tau)+A_{22}(\tau)T_{vv}(\tau) + A_1(\tau)T_u(\tau)+A_2(\tau)T_v(\tau) + A(\tau)T(\tau)+Q(\tau)]\delta T(t-\tau)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau + \int_0^t \iint_\Omega [R(\tau)\delta T_\tau(\tau)+A_{11}(\tau)\delta T_{uu}(\tau) + A_{12}(\tau)\delta T_{uv}(\tau)+A_{22}(\tau)\delta T_{vv}(\tau) + A_1(\tau)\delta T_u(\tau)+A_2(\tau)\delta T_v(\tau) + A(\tau)\delta T(\tau)]T(t-\tau)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau. \tag{10}$$

Using the integration formula by parts, the Green formula and the commutative property of the convolution, the terms entering into the second integral in formula (10) will be transformed in the following way:

$$\int_0^t \iint_\Omega R(\tau)\delta T_\tau(\tau)T(t-\tau)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau = \iint_\Omega [R(t)\delta T(t)T(0)-R(0)\delta T(0)T(t)]\,\mathrm{d}u\,\mathrm{d}v + \int_0^t \iint_\Omega [R_\tau(t-\tau)T(\tau) + R(t-\tau)T_\tau(\tau)]\delta T(t-\tau)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau \tag{11}$$

$$\int_0^t \iint_\Omega A_{11}(\tau)\delta T_{uu}(\tau)T(t-\tau)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau = \int_0^t \int_L A_{11}(\tau)\delta T_u(\tau)T(t-\tau)n_1\,\mathrm{d}l\,\mathrm{d}\tau - \int_0^t \iint_\Omega [A_{11u}(\tau)T(t-\tau) + A_{11}(\tau)T_u(t-\tau)]\delta T_u(\tau)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau = \int_0^t \int_L [A_{11}(\tau)T(t-\tau)\delta T_u(\tau) - A_{11u}(\tau)T(t-\tau)\delta T(\tau) - A_{11}(\tau)T_u(t-\tau)\delta T(\tau)]n_1\,\mathrm{d}l\,\mathrm{d}\tau + \int_0^t \iint_\Omega [A_{11uu}(\tau)T(t-\tau) + 2A_{11u}(\tau)T_u(t-\tau)+A_{11}(\tau)T_{uu}(t-\tau)] \times \delta T(\tau)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau \tag{12}$$

$$\int_0^t \iint_\Omega A_{12}(\tau)\delta T_{uv}(\tau)T(t-\tau)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau = \int_0^t \int_L [A_{12}(\tau)T(t-\tau)\delta T_u(\tau)n_2 - (A_{12v}(\tau)T(t-\tau)+A_{12}(\tau)T_v(t-\tau)) \times \delta T(\tau)n_1]\,\mathrm{d}l\,\mathrm{d}\tau + \int_0^t \iint_\Omega [A_{12uv}(\tau)T(t-\tau) + A_{12u}(\tau)T_v(t-\tau)+A_{12v}(\tau)T_u(t-\tau) + A_{12}(\tau)T_{uv}(t-\tau)]\delta T(\tau)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau \tag{13}$$

$$\int_0^t \iint_\Omega A_{22}(\tau)\delta T_{vv}(\tau)T(t-\tau)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau = \int_0^t \int_L [A_{22}(\tau)T(t-\tau)\delta T_v(\tau) - (A_{22}(\tau)T_v(t-\tau)+A_{22v}(\tau)T(t-\tau)) \times \delta T(\tau)]n_2\,\mathrm{d}l\,\mathrm{d}\tau + \int_0^t \iint_\Omega [A_{22vv}(\tau)T(t-\tau) + 2A_{22v}(\tau)T_v(t-\tau)+A_{22}(\tau)T_{vv}(t-\tau)] \times \delta T(\tau)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau \tag{14}$$

$$\int_0^t \iint_\Omega A_1(\tau)\delta T_u(\tau)T(t-\tau)\,du\,dv\,d\tau = \int_0^t \int_L A_1(\tau)T(t-\tau)\delta T(\tau)n_1\,dl\,d\tau - \int_0^t \iint_\Omega [A_{1u}(\tau)T(t-\tau) + A_1(\tau)T_u(t-\tau)]\delta T(\tau)\,du\,dv\,d\tau \quad (15)$$

$$\int_0^t \iint_\Omega A_2(\tau)\delta T_v(\tau)T(t-\tau)\,du\,dv\,d\tau = \int_0^t \int_L A_2(\tau)\delta T(\tau)T(t-\tau)n_2\,dl\,d\tau - \int_0^t \iint_\Omega [A_{2v}(\tau)T(t-\tau) + A_2(\tau)T_v(t-\tau)]\delta T(\tau)\,du\,dv\,d\tau. \quad (16)$$

In formulae (12)–(16) n_1 and n_2 are the direction cosines of the normal to curve L. Taking into account formulae (11)–(16) and again using the commutative property of the convolution, it is not difficult to write formula (10) in the form

$$\begin{aligned}\delta J_1 = &\int_0^t \iint_\Omega \{[R(\tau)+R(t-\tau)]T_\tau(\tau) + [A_{11}(\tau)+A_{11}(t-\tau)]T_{uu}(\tau) + [A_{12}(\tau)+A_{12}(t-\tau)]T_{uv}(\tau) + [A_{22}(\tau)+A_{22}(t-\tau)]T_{vv}(\tau) \\ &+ [A_1(\tau)+2A_{11u}(t-\tau)+A_{12v}(t-\tau) - A_1(t-\tau)]T_u(\tau) + [A_2(\tau)+A_{12u}(t-\tau) + 2A_{22v}(t-\tau)-A_2(t-\tau)]T_v(\tau)+Q(\tau) \\ &+ [A(\tau)+R_\tau(t-\tau)+A_{11uu}(t-\tau) + A_{12uv}(t-\tau)+A_{22vv}(t-\tau)-A_{1u}(t-\tau) - A_{2v}(t-\tau)+A(t-\tau)]T(\tau)\}\delta T(t-\tau) \\ &\times du\,dv\,d\tau + \iint_\Omega [R(t)T(0)\delta T(t) - R(0)T(t)\delta T(0)]\,du\,dv \\ &+ \int_0^t \int_L [p_1(t-\tau)T(\tau)\delta T_u(t-\tau) + p_2(t-\tau)T(\tau)\delta T_v(t-\tau) + p_3(t-\tau)T(\tau)\delta T(t-\tau) \\ &+ p_4(t-\tau)T_u(\tau)\delta T(t-\tau) + p_5(t-\tau)T_v(\tau)\delta T(t-\tau)]\,dl\,d\tau. \quad (17)\end{aligned}$$

Here, $p_i, i = 1, \ldots, 5$ denote the functions

$$\begin{aligned}p_1(t-\tau) &= A_{11}(t-\tau)n_1 + A_{12}(t-\tau)n_2 \\ p_2(t-\tau) &= A_{22}(t-\tau)n_2 \\ p_3(t-\tau) &= -A_{11u}(t-\tau) - A_{12v}(t-\tau)n_1 - A_{22v}(t-\tau)n_2 + A_1(t-\tau)n_1 + A_2(t-\tau)n_2 \\ p_4(t-\tau) &= -A_{11}(t-\tau)n_1 \\ p_5(t-\tau) &= -A_{11}(t-\tau)n_1 - A_{22}(t-\tau)n_2. \quad (18)\end{aligned}$$

Further, we have in succession

$$\delta J_2 = \iint_\Omega \{R(t)T(t)\delta T(0) + R(t)[T(0)-2T_0]\delta T(t)\}\,du\,dv \quad (19)$$

$$\begin{aligned}\delta J_3 = &\int_0^t \int_L \{q_1(\tau)\delta T(t-\tau) + q_2(\tau)\delta T_u(t-\tau) + q_3(\tau)\delta T_v(t-\tau)] \times [T(\tau) - T_w(\tau)] + [q_1(\tau)T(t-\tau) \\ &+ q_2(\tau)T_u(t-\tau)+q_3(\tau)T_v(t-\tau)] \times \delta T(\tau) + q_4(\tau)\delta T(\tau)T_u(t-\tau) + q_4(\tau)T(\tau)\delta T_u(t-\tau) + q_5(\tau) \\ &\times \delta T(\tau)T_v(t-\tau) + q_5(\tau)T(\tau) \times \delta T_v(t-\tau)\}\,dl\,d\tau. \quad (20)\end{aligned}$$

Formulae (17), (19) and (20) yield

$$\begin{aligned}\delta J(T) = &\int_0^t \iint_\Omega \{[R(\tau)+R(t-\tau)]T_\tau(\tau) + [A_{11}(\tau)+A_{11}(t-\tau)]T_{uu}(\tau) + [A_{12}(\tau)+A_{12}(t-\tau)]T_{uv}(\tau) + [A_{22}(\tau)+A_{22}(t-\tau)]T_{vv}(\tau) \\ &+ [A_1(\tau)+2A_{11u}(t-\tau) + A_{12v}(t-\tau)-A_1(t-\tau)]T_u(\tau) + [A_2(\tau)+A_{12u}(t-\tau)+2A_{22v}(t-\tau) - A_2(t-\tau)]T_v(\tau)+Q(\tau) \\ &+ [A(\tau)+R_\tau(t-\tau)+A_{11uu}(t-\tau) + A_{12uv}(t-\tau)+A_{22vv}(t-\tau) - A_{1u}(t-\tau)-A_{2v}(t-\tau)+A(t-\tau)] \times T(\tau)\}\delta T(t-\tau)\,du\,dv\,d\tau \\ &+ \iint_\Omega \{2R(t)[T(0)-T_0]\delta T(t) + [R(t)-R(0)]T(t)\delta T(0)\}\,du\,dv \\ &+ \int_0^t \int_L \{[q_1(\tau)\delta T(t-\tau) + q_2(\tau)\delta T_u(t-\tau)+q_3(\tau)\delta T_v(t-\tau)] \times [T(\tau)-T_w(\tau)] + [(p_3(t-\tau) \\ &+ q_1(t-\tau))T(\tau) + (p_4(t-\tau)\end{aligned}$$

$$+q_2(t-\tau)+q_4(t-\tau))T_u(\tau)$$
$$+(p_5(t-\tau)+q_3(t-\tau)$$
$$+q_5(t-\tau))T_v(\tau)]\delta T(t-\tau)$$
$$+[p_1(t-\tau)+q_4(\tau)]T(\tau)\delta T_u(t-\tau)$$
$$+[p_2(t-\tau)+q_5(\tau)]T(\tau)$$
$$\times \delta T_v(t-\tau)\}\,\mathrm{d}l\,\mathrm{d}\tau. \qquad (21)$$

Define the unknown functions $q_i, R, A_i, A_{ij}, A, Q$ by the equalities

$$R(t) = R(0)$$
$$p_3(\tau)+q_1(\tau) = 0$$
$$p_4(\tau)+q_2(\tau)+q_4(\tau) = 0$$
$$p_5(\tau)+q_3(\tau)+q_5(\tau) = 0$$
$$p_1(t-\tau)+q_4(\tau) = 0$$
$$p_2(t-\tau)+q_5(\tau) = 0 \qquad (22)$$

and, when $0 \leqslant \tau \leqslant 1/2t$, by the relations

$$R(\tau)+R(t-\tau) = 2c\rho(\tau),$$
$$A_{ij}(\tau)+A_{ij}(t-\tau) = -2a_{ij}(\tau), \quad i,j = 1,2$$
$$A_1(\tau)+2A_{11u}(t-\tau)+A_{12v}(t-\tau)-A_1(t-\tau) = -2a_1(\tau),$$
$$A_2(\tau)+2A_{22v}(t-\tau)+A_{12u}(t-\tau)-A_2(t-\tau) = -2a_2(\tau),$$
$$A(\tau)+R_\tau(t-\tau)+A_{11uu}(t-\tau)+A_{12uv}(t-\tau)+A_{22vv}(t-\tau)-A_{1u}(t-\tau)-A_{2v}(t-\tau)+A(t-\tau) = 0,$$
$$Q(\tau) = -2q(\tau). \qquad (23)$$

It is obvious that the system of equations (23) is consistent. Then, it follows from formulae (18) and (22) that the functions $q_i(\tau), i = 1,\dots,5$ have been determined. The consideration of formulae (18) and (21)–(23) shows that when $0 \leqslant \tau \leqslant 1/2t$, the function $T(u,v,\tau)$, which makes the first variation of the functional $J(T)$ vanish ($\delta J(T) = 0$), is the solution of the initial problem (6)–(8).

DERIVATION OF THE SYSTEM OF EULER EQUATIONS WITH THE USE OF THE GALERKIN–KANTOROVICH METHOD FOR AN APPROXIMATE SOLUTION

Let the functions $\phi^{ij}(u,v), i,j = 0,1,2,\dots$, form the full system of functions in region Ω. The approximate solution of the problem (6)–(8) will be sought in the form

$$T_{nm}(u,v,\tau) = \sum_{i=0}^{n}\sum_{j=0}^{m} V_{ij}(\tau)\phi^{ij}(u,v). \qquad (24)$$

The unknown functions $V_{ij}(\tau), i,j = 0,1,2,\dots$, are selected from the condition that the function T_{nm}, determined by formula (24), transforms the first variation of the functional $J(T)$ into zero ($\delta J(T_{nm}) = 0$). Formulae (21)–(24) yield

$$\delta J(T_{nm}) = \sum_{k,l}\int_0^t\iint_\Omega \sum_{i,j}[(R(\tau)+R(t-\tau))V'_{ij}(\tau)\phi^{ij}+((A_{11}(\tau)+A_{11}(t-\tau))\phi^{ij}_{uu}+(A_{12}(\tau)+A_{12}(t-\tau))\times\phi^{ij}_{uv}+(A_{22}(\tau)+A_{22}(t-\tau))\phi^{ij}_{vv}+(A_1(\tau)+2A_{11u}(t-\tau)+A_{12v}(t-\tau)-A_1(t-\tau))\times\phi^{ij}_u+(A_2(\tau)+A_{12u}(t-\tau)+2A_{22v}(t-\tau)-A_2(t-\tau))\phi^{ij}_v+(A(\tau)+R_\tau(t-\tau)+A_{11uu}(t-\tau)+A_{12uv}(t-\tau)+A_{22vv}(t-\tau)-A_{1u}(t-\tau)-A_{2v}(t-\tau)+A(t-\tau))\times\phi^{ij})V_{ij}(\tau)]+Q(\tau)\}\phi^{kl}\delta V_{kl}(t-\tau)\times\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tau + 2\sum_{k,l}\iint_\Omega R(t)\left[\sum_{i,j}V_{ij}(0)\times\phi^{ij}-T_0\right]\phi^{kl}\delta V_{kl}(t)\,\mathrm{d}u\,\mathrm{d}v + \sum_{k,l}\int_0^t\int_L\left[\sum_{i,j}V_{ij}(\tau)\phi^{ij}-T_w(\tau)\right]\times[q_1(\tau)\phi^{kl}+q_2(\tau)\phi^{kl}_u+q_3(\tau)\phi^{kl}_v]\times\delta V_{kl}(t-\tau)\,\mathrm{d}l\,\mathrm{d}\tau = 0. \qquad (25)$$

Introduce the following notation:

$$\iint_\Omega[R(\tau)+R(t-\tau)]\phi^{ij}\phi^{kl}\,\mathrm{d}u\,\mathrm{d}v = \alpha_{ijkl}(\tau),$$

$$\iint_\Omega\{[A_{11}(\tau)+A_{11}(t-\tau)]\phi^{ij}_{uu}+[A_{12}(\tau)+A_{12}(t-\tau)]\phi^{ij}_{uv}+[A_{22}(\tau)+A_{22}(t-\tau)]\phi^{ij}_{vv}+[A_1(\tau)+2A_{11u}(t-\tau)+A_{12v}(t-\tau)-A_1(t-\tau)]\phi^{ij}_u+[A_2(\tau)+A_{12u}(t-\tau)+2A_{22v}(t-\tau)-A_2(t-\tau)]\phi^{ij}_v+[A(\tau)+R_\tau(t-\tau)+A_{11uu}(t-\tau)+A_{12uv}(t-\tau)+A_{22vv}(t-\tau)-A_{1u}(t-\tau)-A_{2v}(t-\tau)+A(t-\tau)]\phi^{ij}\}\phi^{kl}\,\mathrm{d}u\,\mathrm{d}v = \beta_{ijkl}(\tau),$$

$$\iint_\Omega R(t)\phi^{ij}\phi^{kl}\,\mathrm{d}u\,\mathrm{d}v = \gamma_{ijkl}(t),$$

$$-\iint_\Omega R(t)T_0\phi^{kl}\,\mathrm{d}u\,\mathrm{d}v = \gamma_{kl}(t),$$

$$\int_L [q_1(\tau)\phi^{kl}+q_2(\tau)\phi^{kl}_u+q_3(\tau)\phi^{kl}_v]\phi^{ij}\,dl = \delta_{ijkl}(\tau),$$

$$-\int_L T_w(\tau)[q_1(\tau)\phi^{kl}+q_2(\tau)\phi^{kl}_u+q_3(\tau)\phi^{kl}_v]\,dl = \alpha_{kl}(\tau),$$

$$\iint_\Omega Q(\tau)\phi^{kl}\,du\,dv = \beta_{kl}(\tau). \quad (26)$$

With the use of formula (26), equality (25) can be written as

$$\delta J(T_{nm}) = \sum_{k,l}\int_0^t \Big[\sum_{i,j}(\alpha_{ijkl}(\tau)V'_{ij}(\tau) + \beta_{ijkl}(\tau)V_{ij}(\tau))+\beta_{kl}(\tau)\Big]\delta V_{kl}(t-\tau)\,d\tau$$
$$+2\sum_{k,l}\Big[\sum_{i,j}\gamma_{ijkl}(t)V_{ij}(0)+\gamma_{kl}(t)\Big]\delta V_{kl}(t)$$
$$+\sum_{k,l}\int_0^t\Big[\sum_{i,j}\delta_{ijkl}(\tau)V_{ij}(\tau)+\alpha_{kl}(\tau)\Big] \times \delta V_{kl}(t-\tau)\,d\tau = 0. \quad (27)$$

It follows from formula (27) that the unknown functions $V_{ij}(\tau)$ represent the solution of the following Cauchy problem for the system of ordinary differential equations

$$\sum_{i,j}\{\alpha_{ijkl}(\tau)V'_{ij}(\tau)+[\beta_{ijkl}(\tau) + \delta_{ijkl}(\tau)]V_{ij}(\tau)\}+\beta_{kl}(\tau)+\alpha_{kl}(\tau) = 0,$$
$$\sum_{i,j}\gamma_{ijkl}(t)V_{ij}(0)+\gamma_{kl}(t) = 0,$$
$$k = 0,1,\ldots,n; \quad l = 0,1,\ldots,m. \quad (28)$$

Thus, the construction of an approximate solution of problem (6)–(8) is reduced to the solution of problem (28).

EXAMPLE OF THE SOLUTION OF A TWO-DIMENSIONAL UNSTEADY-STATE HEAT CONDUCTION PROBLEM IN THE REGION WITH A MOVING BOUNDARY

As an example of the application of the above method, consider the formation in time of a temperature field in a deformable prism which initially had a circular cross-section. Assume the deformation to be such that the circle of radius $a(0)$ changes with time to an ellipse with the same cross-sectional area, so that the region Q_t is

$$Q_t = \left\{(x,y,\tau): \frac{x^2}{a^2(\tau)}+a^2(\tau)y^2 < 1, \quad \tau\in(0,t)\right\},$$

where $a(\tau)$ is the value of the ellipse's small half-axis which is an arbitrary positive function of time.

Let the initial temperature T_0 be the same everywhere throughout the prism and its bounding surface be maintained at a time constant and everywhere the same temperature T_w. The thermophysical properties of the body material will also be assumed constant. The power of the volumetric heat generating sources will be assumed constant in time and space.

Then, remembering that hereafter will everywhere represent the argument $\lambda\tau/c\rho$, the following boundary-value problem is arrived at in the region Q_t

$$T_\tau = \operatorname{div}\nabla T+q/\lambda, \quad (x,y,\tau)\in Q_t \quad (29)$$

$$T(x,y,0) = T_0, \quad (x,y)\in\Omega(0) \quad (30)$$

$$T(x,y,\tau) = T_w, \quad (x,y,\tau)\in S_t. \quad (31)$$

Assume that the function $a(\tau)$ has a continuous first-order derivative and introduce new variables

$$u = x/a(\tau), \quad v = ya(\tau), \quad \tau = \tau \quad (32)$$

so that problem (29)–(31) will be written in the form

$$T_\tau = \frac{1}{a^2}T_{uu}+a^2T_{vv}+\frac{a'}{a}(uT_u-vT_v)+q/\lambda, \quad (u,v,\tau)\in V_t \quad (33)$$

$$T(u,v,0) = T_0, \quad (u,v)\in\Omega \quad (34)$$

$$T(u,v,\tau) = T_w, \quad (u,v,\tau)\in F_t. \quad (35)$$

Here

$$V_t = \{(u,v,\tau):(u,v)\in\Omega, \quad \tau\in(0,t)\},$$
$$\Omega = \{(u,v):u^2+v^2 < 1\},$$
$$F_t = \{(u,v,\tau):u^2+v^2 = 1, \quad \tau\in(0,t)\}.$$

By not limiting the generality, it is possible to assume that the constant $T_w = 0$, since the function $T-T_w$ is the solution of equation (33). An approximate solution of problem (33)–(35) will be sought in the form

$$T_{nm}(u,v,\tau) = \sum_{i=0}^{n}\sum_{j=0}^{m} V_{ij}(\tau)u^iv^j. \quad (36)$$

It is clear that the solution $T(u,v,\tau)$ of problem (33)–(35) is an even function with respect to variables u and v. Therefore, it will be required that the function $T_{nm}(u,v,\tau)$, determined by formula (36), be also an even one in variables u and v and, moreover, could vanish on the surface F_t. It follows from the condition of evenness $(T_{nm}(u,v,\tau) = T_{nm}(-u,v,\tau) = T_{nm}(u,-v,\tau))$ that

$$T_{nm}(u,v,\tau) = \sum_{i=0}^{n}\sum_{j=0}^{m} V_{ij}(\tau)u^{2i}v^{2j} \quad (37)$$

and from the condition $T_{nm}(u,v,\tau) = 0$ at $u^2+v^2 = 1$, that

$$T_{nm}(u,v,\tau) = \sum_{i=1}^{n}\sum_{j=0}^{m} V_{ij}(\tau)(1-u^2-v^2)^iv^{2j}. \quad (38)$$

It is clear that the functions $V_{ij}(\tau)$ in formulae (36)–(38) are different. So, the approximate solution of problem (33)–(35) is

$$T_{nm}(u,v,\tau) = \sum_{i=1}^{n}\sum_{j=0}^{m} V_{ij}(\tau)\phi^{ij} \quad (39)$$

where the coordinate functions

$$\phi^{ij} = (1-u^2-v^2)^i v^{2j}.$$

Since in the example considered

$$c\rho(\tau) = 1, \qquad a_{11}(\tau) = \frac{1}{a^2(\tau)},$$

$$a_{12}(\tau) = 0, \qquad a_{22}(\tau) = a^2(\tau),$$

$$a_1(\tau) = \frac{a'(\tau)}{a(\tau)}u, \quad a_2(\tau) = -\frac{a'(\tau)}{a(\tau)}v,$$

$$Q(\tau) = -2q/\lambda,$$

then, based on equation (23), it follows that formulae (26) for $0 \leqslant \tau \leqslant 1/2t$ will be written in the form

$$\alpha_{ijkl} = 2\gamma_{ijkl} = 2\iint_{\Omega} \phi^{ij}\phi^{kl}\,du\,dv,$$

$$\alpha_{kl} = \delta_{ijkl} = 0,$$

$$\gamma_{kl} = -T_0 \iint_{\Omega} \phi^{kl}\,du\,dv,$$

$$\beta_{kl} = -2q/\lambda \iint_{\Omega} \phi^{kl}\,du\,dv,$$

$$\beta_{ijkl}(\tau) = \iint_{\Omega}\left[-\frac{2}{a^2(\tau)}\phi^{ij}_{uu} - 2a^2(\tau)\phi^{ij}_{vv} - 2\frac{a'(\tau)}{a(\tau)}u\phi^{ij}_{u} + 2\frac{a'(\tau)}{a(\tau)}v\phi^{ij}_{v}\right]\phi^{kl}\,du\,dv. \quad (40)$$

Problem (28) for the determination of the unknown functions $V_{ij}(\tau)$ of expansion (39) will acquire the form

$$\begin{cases} \sum_{i,j}[\alpha_{ijkl}V'_{ij}(\tau) + \beta_{ijkl}(\tau)V_{ij}(\tau)] + \beta_{kl} = 0, \\ \sum_{i,j}\alpha_{ijkl}V_{ij}(0) + 2\gamma_{kl} = 0, \\ \qquad i,k = 1,2,\ldots,n; \quad j,l = 0,1,\ldots,m. \end{cases} \quad (41)$$

Calculate the approximate first-order solution $T_{10} = V_{10}(\tau)\phi^{10}(u,v) = V_{10}(\tau)(1-u^2-v^2)$. It follows from equation (41) that this solution can be determined from the following problem

$$\begin{cases} \alpha_{1010}V'_{10}(\tau) + \beta_{1010}(\tau)V_{10}(\tau) + \beta_{10} = 0, \\ \alpha_{1010}V_{10}(0) + 2\gamma_{10} = 0, \end{cases}$$

whence

$$V_{10}(\tau) = \left[-\frac{2\gamma_{10}}{\alpha_{1010}} - \frac{\beta_{10}}{\alpha_{1010}} \times \int_0^{\tau} \exp\left(\frac{1}{\alpha_{1010}}\int_0^{\eta}\beta_{1010}(\xi)\,d\xi\right)d\eta\right] \times \exp\left(-\frac{1}{\alpha_{1010}}\int_0^{\tau}\beta_{1010}(\eta)\,d\eta\right). \quad (42)$$

Formula (40) gives

$$\alpha_{1010} = \frac{2}{3}\pi, \quad \beta_{10} = -\pi q/\lambda, \quad \gamma_{10} = -\frac{\pi}{2}T_0,$$

$$\beta_{1010}(\tau) = \iint_{\Omega}(1-u^2-v^2)\left[\frac{4}{a^2(\tau)} + 4a^2(\tau) + 4\frac{a'(\tau)}{a(\tau)}u^2 - 4\frac{a'(\tau)}{a(\tau)}v^2\right]du\,dv = 2\pi\frac{1+a^4(\tau)}{a^2(\tau)}. \quad (43)$$

It follows from formulae (42) and (43) that the first approximation is calculated from

$$T_{10}(\tau) = \left[\frac{3}{2}T_0 + \frac{3}{2}\frac{q}{\lambda} \times \int_0^{\tau}\exp\left(3\int_0^{\eta}\frac{1+a^4(\xi)}{a^2(\xi)}d\xi\right)d\eta\right] \times \left[\exp\left(-3\int_0^{\tau}\frac{1+a^4(\eta)}{a^2(\eta}d\eta\right)\right](1-u^2-v^2).$$

In conclusion, a system of equations will be given which determines the approximate second-order solution. The second approximation is prescribed by

$$T_{21}(u,v,\tau) = V_{10}(\tau)\phi^{10} + V_{11}\phi^{11} + V_{20}\phi^{20} = V_{10}(\tau)(1-u^2-v^2) + V_{11}(\tau)(1-u^2-v^2)v^2 + V_{20}(\tau)(1-u^2-v^2)^2.$$

With the use of formulae (40) and (41), simple calculations give that the unknown functions $V_{10}(\tau)$, $V_{11}(\tau)$ and $V_{20}(\tau)$ present the solution of the following Cauchy problem

$$\frac{2}{3}V'_{10}(\tau) + \frac{1}{12}V'_{11}(\tau) + \frac{1}{2}V'_{20}(\tau) + 2\left(\frac{1}{a^2(\tau)} + a^2(\tau)\right)V_{10}(\tau) + \left[\frac{1}{3}\left(\frac{1}{a^2} + a^2\right) + \frac{1}{12}\frac{a'}{a}\right]V_{11}(\tau) + 4\left(\frac{1}{a^2} + a^2\right)V_{20}(\tau) = q/\lambda$$

$$\frac{1}{12}V'_{10}(\tau)+\frac{1}{40}V'_{11}(\tau)+\frac{1}{20}V'_{20}(\tau)+\left[\frac{1}{3}\left(\frac{1}{a^2}+a^2\right)-\frac{1}{12}\frac{a'}{a}\right]V_{10}(\tau)+\left(\frac{1}{8a^2}+\frac{3}{8}a^2\right)V_{11}(\tau)+\left(\frac{476}{735a^2}+\frac{1148}{2255}a^2+\frac{611}{711}\frac{a'}{a}\right)V_{20}(\tau)=\frac{q}{6\lambda}$$

$$\frac{1}{2}V'_{10}(\tau)+\frac{1}{20}V'_{11}(\tau)+\frac{2}{5}V'_{20}(\tau)+\frac{4}{3}\left(\frac{1}{a^2}+a^2\right)V_{10}(\tau)+\left(\frac{1}{6a^2}-\frac{5}{6}a^2+\frac{1}{15}\frac{a'}{a}\right)V_{11}(\tau)+\frac{17}{3}\left(\frac{1}{a^2}+a^2\right)V_{20}(\tau)=\frac{8q}{15\lambda}$$

$$\frac{2}{3}V_{10}(0)+\frac{1}{12}V_{11}(0)+\frac{1}{2}V_{20}(0)=T_0$$

$$\frac{1}{12}V_{10}(0)+\frac{1}{40}V_{11}(0)+\frac{1}{20}V_{20}(0)=\frac{1}{6}T_0$$

$$\frac{1}{2}V_{10}(0)+\frac{1}{20}V_{11}(0)+\frac{2}{5}V_{20}(0)=\frac{2}{3}T_0.$$

CONCLUSIONS

The solution of the unsteady-state heat conduction problems for a plane region with the outer boundary moving arbitrarily in time can be performed successfully on the basis of the variational description which uses the convolution-type functional, after preliminary transition to the cylindrical region. In this case approximate analytical or numerical solution of the problem stated can be obtained by applying, for example, the Galerkin–Kantorovich method. It should also be noted that the method developed can also be extended, without great changes of the convolution-type functional, to the case of the second- and third-kind boundary conditions by assigning the heat flux density or the linear coupling between the temperature and its gradient on the body surface.

REFERENCES

1. E. M. Kartashov and B. Ya. Lyubov, Analytical methods of the solution of boundary-value heat conduction problems in the region with moving boundaries. A review, *Izv. Akad. Nauk SSSR, Ser. Energ. Transp.* No. 6, 83–111 (1974).
2. B. Ya. Lyubov, Solution of the unsteady-state one-dimensional heat conduction problem for the regions with a uniformly moving boundary, *Dokl. Akad. Nauk SSSR* **57**(6), 551–554 (1974).
3. G. A. Grinberg, Concerning the temperature or concentration fields created inside of an infinite or finite region by moving surfaces on which the time behaviour of temperature or concentration is prescribed, *Prikl. Mat. Mekh.* **33**(6), 1051–1060 (1969).
4. E. M. Kartashov and V. M. Nechayev, The method of Green's functions in the solution of boundary-value problems of the equation of heat conduction in non-cylindrical regions, *Z. Angew. Math. Mech.* **58**, 199–208 (1978).
5. D. G. Wilson, A. D. Solomon and P. T. Boggs (Editors), *Moving Boundary Problems*. Academic Press, New York (1978).
6. N. M. Tsirelman, Variational solution of the unsteady-state heat conduction problem for the regions with a moving boundary, *Teplofiz. Vysok. Temp.* **18**(4), 886–888 (1980).

SOLUTION D'UN PROBLEME DE CONDUCTION THERMIQUE VARIABLE POUR UN MILIEU BIDIMENSIONNEL AVEC UNE FRONTIERE MOBILE

Résumé—A l'aide d'une fonctionnelle de type convolution, on donne une description variationnelle de la conduction de la chaleur avec des conditions aux limites de première espèce pour un milieu bidimensionnel dont la frontière se déplace dans le temps suivant une loi arbitraire. A partir de la méthode Galerkin–Kantorovich, un système correspondant des équations d'Euler est décrit et la solution (numérique ou analytique) détermine le champ de température dans chaque cas particulier. On donne comme exemple les approximations analytiques première et seconde de la solution d'un problème de déformation d'un prisme ayant initialement une section droite circulaire.

LÖSUNG DES INSTATIONÄREN WÄRMELEITPROBLEMS FÜR EIN ZWEIDIMENSIONALES GEBIET MIT BEWEGLICHER BEGRENZUNG

Zusammenfassung—Unter Verwendung des Faltungs-Funktionals wird eine Variationsbeschreibung der instationären Wärmeleitung mit Randbedingungen erster Art für ein zweidimensionales Gebiet vorgestellt, dessen Berandung sich zeitlich verändert. Mit der Galerkin–Kantorovich-Methode wird ein System von Euler-Gleichungen formuliert, deren Lösung (numerisch oder analytisch) zur Bestimmung des Temperaturfeldes in jedem Spezialfall gebraucht wird. In einem Beispiel wird die erste und zweite analytische Näherung an die Lösung des oben geschilderten Problems für den Fall der Deformation eines Prismas mit anfänglich kreisförmiger Querschnittsfläche ermittelt.

РЕШЕНИЕ ЗАДАЧИ НЕСТАЦИОНАРНОЙ ТЕПЛОПРОВОДНОСТИ ДЛЯ ДВУХМЕРНОЙ ОБЛАСТИ С ПОДВИЖНОЙ ГРАНИЦЕЙ

Аннотация—С использованием функционала типа свертки построено вариационное описание процесса нестационарной теплопроводности с граничными условиями первого рода для двухмерной области, граница которой движется во времени по известному произвольному закону. Основываясь на метод Галеркина–Канторовича, выписана соответствующая система уравнений Эйлера, решение которой (численное или аналитическое) необходимо для определения температурного поля в каждом конкретном случае. Дан пример получения аналитических первого и второго приближения к решению сформулированной выше задачи при деформации призмы с первоначально круговым поперечным сечением.